On oscillator–photon resonances



2001 J. Phys. A: Math. Gen. 34 7757

(http://iopscience.iop.org/0305-4470/34/38/304)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 7757–7768 PII: S0305-4470(01)23452-0

# On oscillator–photon resonances

**C Billionnet**

Centre de Physique Théorique, Ecole Polytechnique, 91128 Palaiseau Cedex, France

E-mail: billionnet@cpht.polytechnique.fr

Received 27 March 2001, in final form 9 July 2001
Published 14 September 2001
Online at stacks.iop.org/JPhysA/34/7757

**Abstract**
Considering a charged three-dimensional harmonic oscillator coupled to the photon field by the usual coupling constant, we show that a qualitative change in the possible states of the system occurs when a length and an energy, characteristic quantities of the oscillator, satisfy a simple relation. The frequency being fixed, oscillator–photon resonances change to oscillator–photon bound states when the length increases.

PACS numbers: 03.65.−w, 11.10.−z, 33.80.−b

## 1. Introduction

A lot of work has been done aimed at describing as rigorously as possible the excited states of matter coupled to radiation. It is of course impossible to quote all the papers devoted to the subject. References [1] to [14] are among some works specially connected to this subject. In [8–12] we introduced a parameter $\mu$ which is proved to be useful in studying resonances in this context, even for large values of the coupling constant.

The aim of this paper is to show a physical situation in which this parameter has a physical meaning and may vary. The dependence of the resonances or bound states on this parameter is described. Some of the resonances we present are not considered, usually.

Of course some of these resonances do correspond to what is known as the excited states of the system which is coupled to the boson. We will not try here to make precise which ones because, surprisingly, this is an intricate question. It will be the subject of a forthcoming paper. Indeed to answer this question we should let the coupling constant $\lambda$ decrease to 0, in order to be able to compare the resonances we obtain to the non-perturbed eigenstates. But then the position of the poles must be considered as functions of two variables, $\lambda$ and $\mu$. It occurs that these functions are in general many-valued, even when $\lambda$ and $\mu$ are real. It is a surprising fact that variations with respect to $\lambda$ and $\mu$ do not always commute and this has the consequence that the labelling of the resonances is a tricky problem. To avoid dealing with that question, in all that follows the coupling constant is held fixed.

  

## 2. Oscillator–photon bound states and resonances

### *2.1. The system, its physical parameters and its Hamiltonian*

Let us consider a three-dimensional harmonic oscillator with charge +1 coupled to the photon field. We suppose that the oscillator is isotropic, has mass $m$ and spring constant $k_r$. The reason why its angular frequency $\omega = \sqrt{k_r/m}$ is not the only parameter we introduce is that we want to be free to also vary the space extension of the stationary states. This extension is measured by $\delta := \sqrt{\hbar/m\omega}$. When $\omega$ is fixed, $\delta^{-1}$ will be the $\mu$ parameter. We recall that the wave functions of the states having energy $n\hbar\omega$ are

$$\varphi_{n_x,n_y,n_z}(x) = (\sqrt{\pi}\delta)^{-3/2}\left(2^{n_x+n_y+n_z}\, n_x!n_y!\, n_z!\right)^{-1/2} \times \mathrm{e}^{-\frac{1}{2}\frac{x^2+y^2+z^2}{\delta^2}}\, H_{n_x}\left(\frac{x}{\delta}\right)H_{n_y}\left(\frac{y}{\delta}\right)H_{n_z}\left(\frac{z}{\delta}\right) \tag{1}$$

where $n = n_x + n_y + n_z$. Let us set $\mathcal{E} := \hbar\omega$. By varying $\mathcal{E}$ and $\delta$, we vary the quantity $\mathcal{E}\delta$ which has the same dimension as $\hbar$.

Let $\mathcal{H} = \mathcal{H}_{\mathrm{osc}} \otimes \mathcal{H}_{\mathrm{phot}}$ be the Hilbert space of the system states. The Hamiltonian for the harmonic oscillator is

$$H_{\mathrm{osc}} = \hbar^2(\boldsymbol{k}^{\mathrm{op}})^2\big/2m + k_r\boldsymbol{r}^2\big/2 = \mathcal{E}(a_x^*a_x + a_y^*a_y + a_z^*a_z + 3/2) \tag{2}$$

where $\boldsymbol{r} \in \mathbb{R}^3$, $\boldsymbol{k} \in \mathbb{R}^3$ and $a_x = 2^{-1/2}i(\delta^{-1}x + \delta\partial_x)$.

The photon field, regularized by an ultraviolet cut-off function $\kappa$, which will be superfluous later on, is

$$\boldsymbol{A}(\boldsymbol{r}) = \left(\frac{\hbar}{(2\pi)^3\,\epsilon_0}\right)^{1/2}\sum_{i=1,2}\int\frac{\kappa(\boldsymbol{k})}{\sqrt{2|\boldsymbol{k}|}}\left(\boldsymbol{\epsilon}_i(\boldsymbol{k})c_i^*(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} + \boldsymbol{\epsilon}_i(\boldsymbol{k})c_i(\boldsymbol{k})\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}}\right)\mathrm{d}\boldsymbol{k} \tag{3}$$

where $\epsilon_i(\boldsymbol{k}) = \epsilon_i(\hat{\boldsymbol{k}})$, $i = 1, 2$, are defined everywhere except for a certain direction $\hat{\boldsymbol{k}}_0$ and form a field of two mutually orthogonal polarization vectors ($\hat{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$).

The coupling with the photon field is supposed to be given by simplifying the Hamiltonian

$$H := H_{\mathrm{osc}} \otimes I + I \otimes H_{\mathrm{phot}} + H_I \tag{4}$$

with $H_I := -\frac{q}{m}\boldsymbol{A}(\boldsymbol{r})\cdot\boldsymbol{p}$. We suppress the constant term in $H_{\mathrm{osc}}$.

The first approximation is the so-called round wave approximation in which the interaction is described only by resonant terms:

$$H_I^{\mathrm{RWA}} = \sqrt{\frac{\alpha}{2\pi}}\,\mathcal{E}\delta\,\sum_{i=1,2}\int_{-\infty}^{\infty}\frac{\kappa(\boldsymbol{k})}{2\sqrt{\pi}\sqrt{|\boldsymbol{k}|}} \times\left([\boldsymbol{a}\cdot\boldsymbol{\epsilon}_{\mathrm{i}}(\boldsymbol{k})]\,c_i^*(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} + [\boldsymbol{a}^*\cdot\boldsymbol{\epsilon}_i(\boldsymbol{k})]\,c_i(\boldsymbol{k})\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\right)\mathrm{d}\boldsymbol{k} \tag{5}$$

where $\boldsymbol{a} = (a_x, a_y, a_z)$, $\boldsymbol{\epsilon}_i(\mathbf{k}) = ((\epsilon_i(\boldsymbol{k}))_x, (\epsilon_i(\boldsymbol{k}))_y, (\epsilon_i(\boldsymbol{k}))_z)$ and $\alpha$ is the fine structure constant.

Let $|0\rangle$ denote the fundamental state of the oscillator and $\Omega$ the vacuum in the photon space. Because of the multiplication by $\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$, the interaction Hamiltonian transforms $|0\rangle \otimes |\boldsymbol{k}\boldsymbol{\epsilon}\rangle$ (the photon has a wave vector $\boldsymbol{k}$ and a polarization $\boldsymbol{\epsilon}$) into a state having components on all states $|n\rangle \otimes \Omega$. So, in this paper, we will make a second approximation, getting rid of the excited states of the oscillator except for the first one.

It must be noted that although the model is simple, obtaining all the resonances without this approximation does not seem easy at all and has not been done up to now. (see [12] for a first step). The proof of the existence, in the simple model that we present now, of a resonance which is usually not considered, except when it is real and under certain conditions, is the first one to the best of our knowledge.

More precisely, let $P_{\leqslant 1}$ be the projector on the space spanned in $\mathcal{H}_{\rm osc} \otimes \mathcal{H}_{\rm phot}$ by the following vectors or subspaces: $|0\rangle \otimes \Omega$, $\{|0\rangle\} \otimes \mathcal{F}_{{\rm phot},1}$, $a_x^*|0\rangle \otimes \Omega$, $a_y^*|0\rangle \otimes \Omega$, $a_z^*|0\rangle \otimes \Omega$. ($\mathcal{F}_{{\rm phot},1}$ is the space of one-photon states.) We consider as our final interaction Hamiltonian

$$H_{1,I} := P_{\leqslant 1}\, H^{\rm RWA} P_{\leqslant 1}. \tag{6}$$

In the limit where the cut-off $\kappa$ is removed, it can then be shown that

$$\begin{aligned} H_{1,I} := \lambda_1\, \mathcal{E} \sum_{i=1,2} (a_x^* \otimes c_i(g_{i,x}(\delta,\cdot)) + a_x \otimes (c_i(g_{i,x}(\delta,\cdot)))^* + a_y^* \otimes c_i(g_{i,y}(\delta,\cdot)) \\ + a_y \otimes (c_i(g_{i,y}(\delta,\cdot)))^* + a_z^* \otimes c_i(g_{i,z}(\delta,\cdot)) + a_z \otimes (c_i(g_{i,z}(\delta,\cdot)))^*)\ P_{\leqslant 1} \end{aligned} \tag{7}$$

where

$$\lambda_1 := \sqrt{\frac{\alpha}{2\pi}} \tag{8}$$

and $\boldsymbol{g}_i(\delta,\boldsymbol{k}) = (g_{i,x}(\delta,\boldsymbol{k}), g_{i,y}(\delta,\boldsymbol{k}), g_{i,z}(\delta,\boldsymbol{k})) = g(\delta,\boldsymbol{k})\, \boldsymbol{\epsilon}_i(\boldsymbol{k})\ \forall \boldsymbol{k}, \hat{\boldsymbol{k}} \neq \hat{\boldsymbol{k}}_0$, with

$$g(\delta,\boldsymbol{k}) = \frac{\delta}{2\sqrt{\pi}\sqrt{|\boldsymbol{k}|}}\ \mathcal{F}\big(|\varphi_{0,0,0}|^2\big)(\boldsymbol{k}) = \frac{\delta^{3/2}\ {\rm e}^{-\frac{1}{2}\boldsymbol{k}^2\delta^2/2}}{2\sqrt{\pi}\sqrt{\delta|\boldsymbol{k}|}}. \tag{9}$$

$\mathcal{F}$ is the Fourier transformation. The cut-off function $\kappa$ is no more useful, due to the exponential decrease of $\varphi_{0,0,0}$.

Through the successive simplifications, the original Hamiltonian (4) has thus been changed to another one which has a simple Friedrichs' type form [15]:

$$H_1 := P_{\leqslant 1}\ (H_{\rm osc} \otimes I) P_{\leqslant 1} + P_{\leqslant 1}\ (I \otimes H_{\rm phot}) P_{\leqslant 1} + H_{1,I} \tag{10}$$

with $H_{1,I}$ given by (7).

Let us now calculate the eigenvalues of (10) when they exist and prove that they manifest themselves as resonances when they do not exist.

### *2.2. Determination of bound states and resonances*

*2.2.1. Existence.* Let us state the result. $|1\rangle_x$ denotes $a_x^*|0\rangle$, and $|\mathbf{1}\rangle := (a_x^*|0\rangle, (a_y^*|0\rangle, (a_z^*|0\rangle)$. $z \to (\langle 1_x|[z - H_1(1)]^{-1}|1_x\rangle)^{-1}$ is analytic in the upper half-plane and can be analytically continued into the lower half-plane across the cut $\mathbb{R}^+$. In this paper, we will consider the function that this continuation defines in the complex plane cut along $\mathbb{R}^-$. (The upper lip of the cut is included.)

**Proposition.** *There exist two complex-valued functions $z_1(\mathcal{E},\delta)$ and $z_0(\mathcal{E},\delta)$, defined for every $\mathcal{E} > 0$ and every $\delta > 0$, which are zeros of the multivalued function*

$$z \to \Big(\langle 1_x|[z - H_1(1)]^{-1}|1_x\rangle\Big)^{-1}. \tag{11}$$

*They are of the form $z_i(\mathcal{E},\delta) = \mathcal{E} s_i(\frac{\hbar}{\mathcal{E}\delta})$, where $s_i(\mu)$ is a zero of the analytic continuation into $\mathbb{C}\backslash\mathbb{R}^-$ from the upper half-plane of*

$$s \to f_{1,\mu}(s) := s - 1 - \frac{\alpha}{3\pi}\int_0^\infty \frac{s'\ {\rm e}^{-\frac{1}{2}(s')^2}}{s - \mu s'}\,{\rm d}s'. \tag{12}$$

(*For $\Im s < 0$, the concrete form of this continuation is $f_{1,\mu,+}$ given by* (30).) $\frac{\alpha}{3\pi} \simeq 7.74 \times 10^{-4}$ *and $\mu_c := \frac{\alpha}{3\sqrt{2\pi}} \simeq 9.7 \times 10^{-4}$ is a transition value for $\frac{\hbar}{\mathcal{E}\delta}$, at which $z_0(\mathcal{E},\delta)$*

*changes from negative values to complex ones in the second sheet through the branch point of* (12) *when* $\frac{\hbar}{\mathcal{E}\delta}$ *increases. The critical relation is thus*

$$\mathcal{E}\,\delta\alpha = \frac{3}{\sqrt{2\pi}}\,h. \tag{13}$$

*More precisely,*

(a) $z_1(\mathcal{E},\delta)$ *is a zero of the continuation of* (11), *or of* $f_{\hbar/(\mathcal{E}\delta)}$, *into the lower half-plane through the positive real axis. It is complex for every value of* $\mathcal{E}$ *and* $\delta$.

($\mathrm{b}_1$) *If* $\mathcal{E}\delta < \frac{3}{\alpha\sqrt{2\pi}}\,h \simeq 164\,h$, $z_0(\mathcal{E},\delta)$ *is another complex zero of the continuation of* (11) *into the same region. The coupled oscillator has a resonance corresponding to that complex point. This resonance never becomes real.*

($\mathrm{b}_2$) *If* $\mathcal{E}\delta > \frac{3}{\alpha\sqrt{2\pi}}\,h \simeq 164\,h$, $z_0(\mathcal{E},\delta)$ *is negative and the coupled oscillator has a bound state with energy* $z_0(\mathcal{E},\delta)$. *The eigenspace is three-dimensional. Eigenvectors (depending on* $\mathcal{E}$ *and* $\delta$) *corresponding to the eigenvalue* $z_0(\mathcal{E},\delta)$ *are*

$$\Psi_{0,\boldsymbol{V}}(\boldsymbol{k}) = \boldsymbol{V}\cdot|\mathbf{1}\rangle\otimes\Omega + \frac{\sqrt{\alpha}}{2\pi\sqrt{2}}\,\mathcal{E}\delta|0\rangle\otimes\frac{\mathrm{e}^{-\frac{1}{2}\boldsymbol{k}^2\delta^2/2}}{\sqrt{|\boldsymbol{k}|}\,(z_0(\mathcal{E},\delta)-\hbar|\boldsymbol{k}|)}\,P(\boldsymbol{k})(\boldsymbol{V}) \tag{14}$$

*with* $\boldsymbol{V}$ *arbitrary in* $\mathbb{C}^3$ *and* $P(\boldsymbol{k})$ *denoting the orthogonal projector on* $\boldsymbol{k}^\perp$. *These states have angular momentum* 1 *as linear superpositions of angular momentum* 1 *states.*

**Remark.** As we have mentioned in section 1, one of the two resonances (or bound states) is the one usually associated with the unstable states corresponding to $|1_x\rangle$, $|1_y\rangle$, $|1_z\rangle$. But the difficulty of knowing which one holds to the fact that the usual calculation and even the definition of the excited states are in term of $\lambda$, and variation with respect to $\mu$ and $\lambda$ do not commute for all paths. This will be discussed in the paper in preparation mentioned in section 1. This question is also linked with the possibility, in case ($\mathrm{b}_1$), of a coincidence between $z_0(\mathcal{E},\delta)$ and $z_1(\mathcal{E},\delta)$ for some special values of $\mathcal{E}\delta$ (and $\alpha$).

$\mathcal{E}$ and $\delta$ are independent quantities. $\mathcal{E}$ being given, if $\delta$ increases, the proposition tells us that there is a qualitative transition in the nature of the possible states of the system when $\mathcal{E}\delta$ crosses $164h$. The critical distance corresponding to 1 eV is $\delta_c \simeq 0.2$ mm. Physically, $\delta$ is large when $k_r$ is small ($\delta = \hbar^{1/2}(mk_r)^{-1/4} = \mathcal{E}^{1/2}\,k_r^{-1/2}$).

**Proof.** The first part of the proof that follows is not essentially a new result; it is only given here to take explicitly into account the vector aspect of the photon, and to follow the dependence in $\mathcal{E}$ and $\delta$ of the oscillator–photon bound states. The second part proves that this bound state turn into a resonance for sufficiently large and arbitrary $\delta$. This result is new.

*Part 1.* Let us look for eigenvectors of (10) of the form

$$\psi(\boldsymbol{k}) := (\boldsymbol{\alpha}\cdot|\mathbf{1}\rangle)\otimes\Omega + |0\rangle\otimes\varphi(\boldsymbol{k})\,\mathrm{Proj}_{\boldsymbol{\epsilon}_1,\boldsymbol{\epsilon}_2}(\boldsymbol{k})(\boldsymbol{\beta}_1,\boldsymbol{\beta}_2) \tag{15}$$

where $\varphi$ is in $L^2(\mathbb{R}^3)$, vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}_i$ are constant vectors in $\mathbb{C}^3$ and

$$\mathrm{Proj}_{\boldsymbol{\epsilon}_1,\boldsymbol{\epsilon}_2}(\boldsymbol{k})(\boldsymbol{\beta}_1,\boldsymbol{\beta}_2) := \mathrm{Proj}_{\boldsymbol{\epsilon}_1(\boldsymbol{k})}(\boldsymbol{\beta}_1)\,\boldsymbol{\epsilon}_1(\boldsymbol{k}) + \mathrm{Proj}_{\boldsymbol{\epsilon}_2}(\boldsymbol{k})(\boldsymbol{\beta}_2)\,\boldsymbol{\epsilon}_2(\boldsymbol{k})$$

$\mathrm{Proj}_{\mathbf{u}}$ being the orthogonal projector on $\boldsymbol{u}$. The set of states of the form (14) is invariant by $H_1$. The relation

$$H_1\psi = z\psi \tag{16}$$

is equivalent to the set of relations

$$(\lambda_1\mathcal{E}\ g_\delta(\boldsymbol{k})\boldsymbol{\alpha} - \varphi(\boldsymbol{k})(z-\overline{h}\boldsymbol{k})\boldsymbol{\beta}_1)\cdot \boldsymbol{\epsilon}_1(\boldsymbol{k}) = 0 \tag{16a}$$

$$(\lambda_1\mathcal{E}\ g_\delta(\boldsymbol{k})\boldsymbol{\alpha} - \varphi(\boldsymbol{k})(z-\overline{h}\boldsymbol{k})\boldsymbol{\beta}_2)\cdot \boldsymbol{\epsilon}_2(\boldsymbol{k}) = 0 \tag{16b}$$

$$\mathcal{E}\ \boldsymbol{\alpha} + \lambda_1\mathcal{E}\int \varphi(\boldsymbol{k})\ g_\delta(\boldsymbol{k})\ \mathrm{Proj}_{\epsilon_1,\epsilon_2}(\boldsymbol{k})(\boldsymbol{\beta}_1,\boldsymbol{\beta}_2)\ \mathrm{d}\boldsymbol{k} = z\ \alpha. \tag{16c}$$

$g_\delta$ is written for $g(\delta,\cdot)$. Let us suppose that the function $\boldsymbol{\epsilon}_1(\cdot)$ is chosen in such a way that its three components $\epsilon_{1,x}(\cdot),\ldots$ are linearly independent, and also in such a way that the three components of $\boldsymbol{\epsilon}_2(\cdot)$ are independent.

From (16a) and (16b), if $z<0$, when $\delta$ is large enough, we get

$$\varphi(\boldsymbol{k}) = \frac{C_1\ g_\delta(\boldsymbol{k})}{z-\overline{h}|\boldsymbol{k}|}. \tag{17}$$

(If $z>0$, $\varphi\notin L^2(\mathbb{R}^3)$. The dependence of $\varphi$ on $\delta$ is understood.) We also get

$$\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = C_1^{-1}\ \lambda_1\mathcal{E}\ \boldsymbol{\alpha}. \tag{18}$$

Relation (16c) then implies

$$L\ \boldsymbol{\alpha} = 0 \tag{19}$$

with

$$L_\mu = (z-\mathcal{E})\ I - (\lambda_1\mathcal{E})^2\ \Gamma \tag{20}$$

where we set

$$\Gamma = \int \frac{(g_\delta(\boldsymbol{k}))^2}{z-\overline{h}|\boldsymbol{k}|}\ P(\boldsymbol{k})\ \mathrm{d}\boldsymbol{k}. \tag{21}$$

$L$ and $\Gamma$ are operators on $\mathbb{R}^3$ depending on the eigenvalue $z$. We have $\int_{S_2} P(\hat{\boldsymbol{k}}) = \frac{8\pi}{3} I$ and thus

$$\Gamma = \frac{2}{3}\left(\delta^2\int_0^\infty \frac{\mathrm{e}^{-\frac{1}{2}\delta^2\boldsymbol{k}^2}}{z-\overline{h}\ |\boldsymbol{k}|}\ |\boldsymbol{k}|\ \mathrm{d}|\boldsymbol{k}|\right) I. \tag{22}$$

Therefore, $L\ \boldsymbol{\alpha} = 0$ has a non-zero solution for $\boldsymbol{\alpha}$ only if $z$ is a zero of

$$f(\lambda_1, z) := z - \mathcal{E} - \frac{2}{3}\lambda_1^2\ \mathcal{E}^2\delta^2\ \int_0^\infty \frac{(g_{1,\delta}(|\boldsymbol{k}|))^2}{z-\overline{h}|\boldsymbol{k}|}\ \mathrm{d}|\boldsymbol{k}| \tag{23}$$

where

$$g_{1,\delta}(\xi) := \xi^{1/2}\ e^{-\frac{1}{2}\xi^2\delta^2/2} \tag{24}$$

is now a function of one variable. ($g(\delta,\boldsymbol{k})$ in (9) was a function of three variables.) We will also be interested in the zeros of the analytic continuation of $f(\lambda_1,\cdot)$ across the positive real axis, which we denote by $f_+(\lambda_1,\cdot)$, since, by

$$\langle 1_x|\ [z-H_1]^{-1}|1_x\rangle^{-1} = f(\lambda_1, z) \tag{25}$$

a relation which will be proved at the end of the section, this continuation gives resonances.

Let us now introduce the dimensionless quantities

$$s = \mathcal{E}^{-1}\ z \qquad \text{and} \qquad \gamma := \frac{\mathcal{E}\delta}{\overline{h}}. \tag{26}$$

We get

$$f(\lambda_1, z) = \mathcal{E}\ f_{1,1/\gamma}(s) \tag{27}$$

with $f_{1,\mu}(s)$ given by (12).

*Part 2.* So we are now left with the determination of zeros of $f_{1,\mu}$, with $\mu = 1/\gamma$. (As expected, they depend only on $\frac{\hbar}{\mathcal{E}\delta}$.) Zeros of functions of this type, with various functions in the numerator of the integrand in (12), were determined in [8–12]. Unfortunately, the function

$$g_1(s) := s^{1/2}\,\mathrm{e}^{-\frac{1}{4}s^2}$$

in (12) does not decrease at infinity, except if $|\arg s| < \pi/4$ or $|\arg -s| < \pi/4$; this prevents us from using the theoretical result in [11] for getting zeros of $f_{1,\mu}$. They can be computed numerically (see section 2.2.2), but the theoretical analysis is also useful.

Let us note that there would have been no problem here if we had considered the hydrogen atom instead of the harmonic oscillator. Indeed the behaviour in $\mathrm{e}^{-ar}$ of the stationary states at large distances would give rise to coupling functions decreasing at infinity (and having poles at finite distances). However, the proof in [11] may be adapted to the present case as explained in the appendix.

This implies that $f_{1,\mu}$ or its analytic continuation has at least two zeros $s_0(\mu)$ and $s_1(\mu)$, the eigenvalues of (10) we are looking for being then

$$z_i(\mathcal{E},\delta) = \mathcal{E}s_i\left(\frac{1}{\gamma}\right) = \mathcal{E}s_i\left(\frac{\hbar}{\mathcal{E}\delta}\right). \tag{28}$$

Moreover, the functions $s_i$, or $z_i$, may be described qualitatively by introducing

$$\mu_c := \frac{\alpha}{6\pi}\,\||s|^{-1}g_1^2(s)\|_1 = \frac{\alpha}{6\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-s^2/2}\,\mathrm{d}s = \frac{\alpha}{3\sqrt{2\pi}} \simeq 9.7\times 10^{-4}. \tag{29}$$

Whereas $z_1(\mathcal{E},\delta)$ is complex for all $\mathcal{E}$, $\delta$ and indicates a resonance, $z_0(\mathcal{E},\delta)$ may be complex or real, indicating a resonance or a bound state, depending on whether $\gamma < 3\sqrt{2\pi}\,\alpha^{-1}$ or $\gamma > 3\sqrt{2\pi}\,\alpha^{-1}$. That $z_1(\mathcal{E},\delta)$ never becomes real is due to the fact that $f_{1,\mu}$ has no real zeros if $\mu > \mu_c$ and only one if $\mu \leqslant \mu_c$, since it is an increasing function. The proposition will be completely proved when we have proved (25).

The proof goes the usual way by considering the projector $Q$ on the one-dimensional space $\{|1_x\rangle \otimes \Omega\}$, and $\bar{Q} = 1 - Q$. When $\Im z > 0$, so that $\bar{Q}[z - H_1]\bar{Q}$ is invertible, the operator $Q(z-\mathcal{E})Q - QH_{1,I}\bar{Q}(\bar{Q}[z-H_1]\bar{Q})^{-1}\bar{Q}H_{1,I}Q$ acting in this space is a multiplication operator with inverse $\langle 1_x|[z - H_1]^{-1}|1_x\rangle I$. To calculate the former, one develops perturbatively $(\bar{Q}[z - H_1]\bar{Q})^{-1}$. Only the term without coupling in $(\bar{Q}[z - H_1]\bar{Q})^{-1}$ gives a non-zero contribution. Indeed, after integration on all directions of the photon momenta, only the diagonal terms $(\epsilon_1)_x(\boldsymbol{k})(\epsilon_1)_x(\boldsymbol{k}) + (\epsilon_2)_x(\boldsymbol{k})(\epsilon_2)_x(\boldsymbol{k})$ (the same with $y$ and $z$) give a non-zero contribution $8\pi/3$. These terms are the matrix elements of $P(\boldsymbol{k})$. □

*2.2.2. Numerical values.* We are particularly interested in the zero $z_0(\mathcal{E},\delta)$.

In case (a) of the proposition, that is for values of $\gamma = \frac{\mathcal{E}\delta}{\hbar}$ below the critical value $\gamma_c := (\mu_c)^{-1} \simeq 1030$, the continuation $f_{1,1/\gamma,+}$ of $f_{1,1/\gamma}$ across the positive real axis from the upper half-plane is

$$f_{1,\frac{1}{\gamma},+} = s - 1 - \frac{\alpha}{3\pi}\int_0^{\infty}\frac{s_1\,\mathrm{e}^{-\frac{1}{2}s_1^2}}{s - s_1/\gamma}\,\mathrm{d}s_1 + \frac{2}{3}\,i\,\alpha\,\gamma^2\,s\,\mathrm{e}^{-\frac{1}{2}\gamma^2 s^2}. \tag{30}$$

The path of the zero $s_0(\frac{1}{\gamma})$ in the complex plane as $\gamma$ varies between 600 and 930 is shown in figure 1. It gives the position of the $z_0$-resonance after multiplication by $\mathcal{E}$. It is difficult to approach the critical value $\gamma_c \simeq 1030$ because of the lack of convergence of the calculation on the computer. However, as expected, it can be seen by refining the calculation that the path is likely to link with the negative real axis at 0 when $\gamma$ increases up to $\gamma_c$, or, if $\mathcal{E}$ is fixed, when $\delta$ increases up to the critical value $\delta_c$.

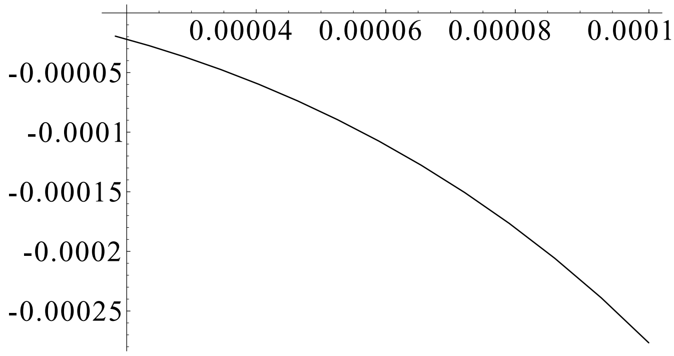


**Figure 1.** Complex values of $s_0 = \mathcal{E}^{-1} z_0(\mathcal{E}, \delta)$ for $600 < \gamma < 930 < \gamma_c$.

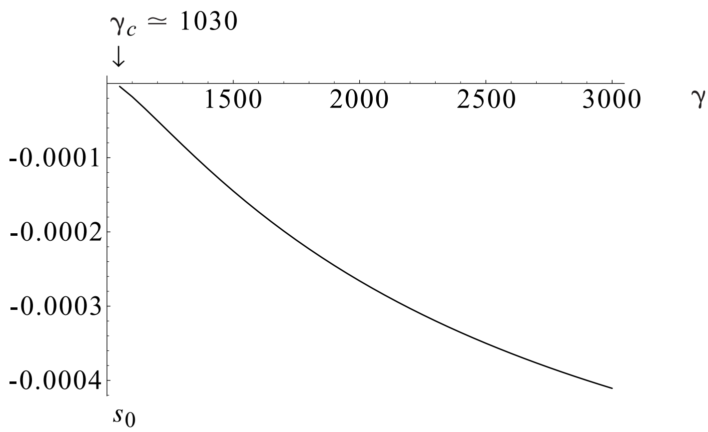


**Figure 2.** Values of $s_0 = \mathcal{E}^{-1} z_0(\mathcal{E}, \delta)$ as a function of $\gamma = \hbar^{-1}\mathcal{E}\delta$ for $\gamma > \gamma_c$.

For $\gamma > \gamma_c$, case (b) of the proposition, the real negative values of $\mathcal{E}^{-1}\, z_0(\mathcal{E}, \delta) = s_0(\frac{1}{\gamma})$ are shown on the vertical axis of the graph in figure 2 as a function of $\gamma$. These are energies of bound states. The following relations can be proved from (12):

$$\lim_{\gamma \to \gamma_c^+} s_0\left(\frac{1}{\gamma}\right) = 0 \qquad \lim_{\gamma \to \infty} s_0\left(\frac{1}{\gamma}\right) \simeq -\frac{\alpha}{3\pi} \simeq -7.7 \times 10^{-4}.$$

The zero $z_1(\mathcal{E}, \delta)$ moves entirely inside the second sheet, when $\gamma$ varies, and

$$\lim_{\gamma \to \infty} z_1\left(\frac{1}{\gamma}\right) \simeq \mathcal{E}\left(1 + \frac{\alpha}{3\pi}\right).$$

## 3. Conclusion, comments and perspectives

### *3.1. Conclusion*

We exhibited two poles of the Hamiltonian resolvent. We left open the question knowing which one is related to the familiar one associated with the unperturbed excited level. The following analogy may help to grasp the difficulty. A second-order algebraic equation has two zeros but when some parameter is introduced in the equation, the two zeros may interchange in some sense, when the parameter varies in the complex plane. In our case, it can be shown that the difficulty even appears in $\mathbb{R}^2$. The important point for us here is that there are two poles.

From the proposition, $z_0$ is real negative when $\mathcal{E}\,\delta\,\alpha$ is sufficiently large. In QED, $\alpha$ is fixed, so the only way to have $\mathcal{E}\,\delta\,\alpha$ large is to have $\mathcal{E}$ large or $\delta$ large (or both). The principal aim of the paper was to present a simple model and its physical parameters which can be used to follow oscillator–photon bound states corresponding to $z_0$. The proof that these states turn into resonances when some characteristics of the oscillator are varied has been given. We do not know of any such statement in the relevant literature.

### *3.2. More complete and more realistic studies of systems interacting with photons*

The model we examined in this paper is certainly a rough one. It would be interesting to improve it, for example, by taking into account the previously neglected excited states of the oscillator, or treating completely the hydrogen atom, or hydrogenic atoms, coupled to the photon field in a non-relativistic way. Although electromagnetic matrix elements given in [16] could be used so that the Hamiltonian is known by its matrix elements on a certain base, it seems to us a difficult task to look for all the resonances. Indeed, even by limiting ourselves to one photon and one excited state, we have seen that the position of the resonances can only be determined by a computer, since no perturbative argument is known at the moment. We have only an existence theorem. Nevertheless, the model we just treated shows that bound states, which would perhaps have been associated with strong coupling, may possibly occur in electromagnetic interactions if the states of the system which couples to the radiation have a sufficiently large extension. In this respect, it would also be interesting to examine the consequences of the result with regard to atoms or large molecules. We showed that even if the bound states are not present, they manifest themselves (theoretically, at least) as resonances. It is expected that to each energy level of the system which is coupled to the photon is associated a countable infinity of bound states or resonances. This has been proved for the complete one-dimensional oscillator coupled to massless scalar bosons via a certain class of meromorphic coupling functions [12].

### *3.3. A possible application to quark–gluon systems*

The coupling constant in the model was the electromagnetic one. Nevertheless, the statement in the proposition still holds with any value of $\alpha$, for instance $\alpha_s$, the coupling constant of strong interactions at low energy. (Let us recall that what we wanted to avoid, for the moment, was to vary $\mu$ and the coupling constant simultaneously. But the coupling constant can in fact be chosen arbitrarily, provided that it is kept fixed in the process of determining the resonances.) Models coupling quark systems in the fundamental state and first excited states to gluons could be examined with the point of view we presented here, neglecting in a first approach spin, flavour and colour degrees of freedom. The possible formation of a $q\bar{q}$-gluon bound state, due to the existence of $z_0$, could be in competition with the bringing of the $q\bar{q}$ system in an upper excited state by the gluon. Since in excited states quarks are likely to be further apart than in the fundamental state, this could yield a confinement mechanism.

## Appendix A

We study here the zeros of (12)

$$f_{1,\mu}(s) := s - 1 - 2\lambda_2^2 \int_0^\infty \frac{s_1\,\mathrm{e}^{-\frac{1}{2}s_1^2}}{s - \mu s_1}\,\mathrm{d}s_1$$

from a theoretical point of view. Numerical values are given in the figures. $\lambda_2$ is $\sqrt{\frac{\alpha}{6\pi}}$. The method in [10, 11] for proving the existence of the zeros $s_0(\mu)$ and $s_1(\mu)$ for any value $\mu_{\max}$ of $\mu$ is to follow the existence of $s_i(\mu)$ as $\mu$ varies on a certain path in $\mathbb{C}$, starting from 0 and ending at $\mu_{\max}$. If it is $s_0(\mu)$ that is followed, the path described by $\mu$ must avoid $\mu_c(\lambda_2)$ (see (29)), for $s_0(\mu)$ to avoid 0, which is the branch point of $f_{1,\mu}$. Near $\mu_c(\lambda_2)$, we suppose that the path follows the half-circle of radius $\epsilon := \mu_c(\lambda_2)\sin\frac{\pi}{12}$ so that $\arg\mu \leqslant \frac{\pi}{12}$. Otherwise, $\mu$ stays real. The path is denoted by $\mathcal{C}_\epsilon(\mu_{\max})$. The path used to follow $s_1(\mu)$ may simply be $[0, \mu_{\max}]$. A local existence theorem on zeros of analytic function is used to perform a step-by-step construction of $s_i(\mu)$, up to a certain limit point $s_{i,\lim}$ for a certain $\mu_{i,\lim}$. The construction is successful if $\mu_{i,\lim}$ is not smaller than $\mu_{\max}$. The way we did this in [11] was to prove the following:

**Proposition A.** *If there is an accumulation point $s_{\lim}$ in the construction, it is necessarily at a finite distance.*

The proof of this assertion in [11] uses the boundedness of $g_1$ at infinity in all complex directions. Although this is not true here, the proposition still remains true.

**Proof.** We are going to prove three lemmas which concern roughly the three regions where $g_1^2$ is respectively exponentially decreasing, exponentially increasing and linearly increasing at infinity. The idea of the proof is to show that $s$ and

$$\varphi(s) := \int_0^\infty \frac{s_1\,\mathrm{e}^{-\frac{s_1^2}{2}}}{s - s_1}\,\mathrm{d}s_1 \tag{A1}$$

have different behaviour at infinity. □

**Lemma A1.** *Set $S_1 := \{\zeta;\ -\frac{\pi}{4} < \arg\zeta \leqslant \pi\}$. $\varphi(\zeta)$ together with its analytic continuation are bounded in $S_1$, for $|\zeta| > 1$.*

**Proof.** (Adaptation of the proof of lemma 1 of [11]). Let $\zeta$ be in $S_1$. Let us consider the disc $D(\zeta, \sin\frac{\pi}{8})$. If $\Im\zeta > 0$ and $D(\zeta, \sin\frac{\pi}{8}) \cap \mathbb{R}^+ = \emptyset$, then $|\varphi(\zeta)| < (\sin\frac{\pi}{8})^{-1}$. If $D(\zeta, \sin\frac{\pi}{8}) \cap \mathbb{R}^+ \neq \emptyset$, then the boundary of $D$ does not intersect $\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\mathbb{R}^+$ and thus $\sup_{z\in\partial D} |g_1(z)|^2 < \infty$. The integral defining $\varphi$ or its continuation $\varphi_+$ may be calculated on a contour the points of which remain at a distance at least $\sin\frac{\pi}{8}$ from $\zeta$, a contour which is either a part of $\mathbb{R}^+$ or a part of $\partial D$. The boundedness follows. Finally, if $-\frac{\pi}{4} < \arg\zeta < 0$ and $D(\zeta, \sin\frac{\pi}{8}) \cap \mathbb{R}^+ = \emptyset$, then $\varphi_+(\zeta)$ is the sum of the principal determination, which is bounded, and the residue part $2\mathrm{i}\pi g_1(\zeta)^2$ which is also bounded. Thus $\varphi$ or $\varphi_+$ are bounded in $S_1$. □

$\varphi$ is also bounded if $|\zeta| > 1$ and $-\pi \leqslant \arg\zeta \leqslant -2\pi/3$. As a consequence, since for $\mu' \in \mathcal{C}_\epsilon(\mu_{\max})$, $s \in S_1$ implies $s/\mu' \in S_1$ or $-\pi < \arg\frac{z}{\mu'} \leqslant -2\pi/3$, the zeros of $f_{1,\mu}(s)$ or $f_{1,\mu,+}(s)$ in $S_1$ form a bounded set.

**Lemma A2.** *For $\theta' > 0$, let $\theta = \frac{\pi}{12} + \theta'$ be in $]0, \frac{\pi}{4}[$. Set $S_\theta := \{\zeta;\ -\frac{3\pi}{4} + \theta \leqslant \arg\zeta \leqslant -\frac{\pi}{4} - \theta\}$. Let $\mu_1, \mu_2$ satisfy $\mu_2 > \mu_1 > 0$. Set $M_{\theta'} := \max\{A_{\theta'}, B\}$ with*

$$A_{\theta'} = \mu_2\left(\frac{2}{\sin 2\theta'}\log\left(\frac{(\mu_2)^2}{4\pi\lambda_2^2}\left(1+\frac{\gamma}{\mu_1}\right)\right)\right)^{1/2} \qquad B = \left(4\sqrt{2}\,\lambda_2^2\gamma\frac{(\mu_2)^2}{\mu_1}\right)^{1/2}. \tag{A2}$$

*Then* $\{|\mu| \in [\mu_1, \mu_2],\ s \in S_\theta,\ |s| > M_{\theta'}\} \Longrightarrow f_{1,\mu,+}(s) \neq 0.$

**Proof.** $s \in S_\theta$ implies $s/\mu \in S_{\theta'}$. We have

$$s^{-1} f_{1,\mu,+}(s) = 1 - \frac{1}{s} - 2\frac{\lambda_2^2}{\mu^2}\left(\frac{s}{\mu}\right)^{-1}\varphi_+\left(\frac{s}{\mu}\right)$$

where

$$\zeta^{-1}\varphi_+(\zeta) = \frac{1}{\zeta}\int_0^\infty \frac{s_1\,\mathrm{e}^{-s_1^2/2}}{\zeta - s_1}\,\mathrm{d}s_1 - 2\mathrm{i}\pi\,\mathrm{e}^{-\frac{1}{2}\zeta^2}.$$

From $\cos(\arg((\frac{s}{\mu})^2)) < -\sin(2\theta')$ and $|s| > A_{\theta'}$ we get

$$\left|\exp\left(-\frac{1}{2}\frac{s^2}{\mu^2}\right)\right| > \exp\left(\frac{1}{2}\left|\frac{s}{\mu}\right|^2\sin(2\theta')\right) > \frac{(\mu_2)^2}{4\pi\lambda_2^2}\left(1+\frac{\gamma}{\mu_1}\right).$$

Besides, $|s| > B$ implies

$$\left|\left(\frac{s}{\mu}\right)^{-1}\int_0^\infty \frac{s_1\,\mathrm{e}^{-\frac{s_1^2}{2}}}{\frac{s}{\mu} - s_1}\,\mathrm{d}s_1\right| < \sqrt{2}\,\frac{(\mu_2)^2}{B^2}.$$

Therefore,

$$2\frac{\lambda_2^2}{\mu^2}\left|\left(\frac{s}{\mu}\right)^{-1}\varphi_+\left(\frac{s}{\mu}\right)\right| > \left(1+\frac{\gamma}{\mu_1}\right) - 2\sqrt{2}\frac{(\mu_2)^2}{(\mu_1)^2}\,\frac{\lambda_2^2}{B^2} = 1 + \frac{\gamma}{2\mu_1}.$$

This implies $|s^{-1} f_{1,\mu,+}(s)| > 0$. □

We are left with proving the boundedness property of the set of zeros in the intermediate region.

**Lemma A3.** *The set of zeros of $f_{1,\mu,+}$ in $\{s;\ \frac{\pi}{4} - \frac{\pi}{12} \leqslant \arg s \leqslant \frac{\pi}{4}\}$, for $|\mu| > \mu_1$ and $\mu \in \mathcal{C}_\epsilon(\mu_{\max})$, is bounded.*

**Proof.** Let us set $t := \frac{s}{\mu(1-i)}$. It can be seen that $f_{1,\mu,+}(t\mu(1-i) = 0$ is equivalent to

$$x(\mu, t) := \mu\, t - 1/2 + 2\frac{\lambda_2^2}{\mu}I_2(t) - 4\pi\frac{\lambda_2^2}{\mu}t\sin(t^2) = 0 \tag{A3a}$$

$$y(\mu, t) := -1/2 - 4\frac{\lambda_2^2}{\mu}tI_1(t) + 2\frac{\lambda_2^2}{\mu}I_2(t) + 4\pi\frac{\lambda_2^2}{\mu}t\cos(t^2) = 0 \tag{A3b}$$

where

$$I_2(t) := \int_0^\infty \frac{s^2\,\mathrm{e}^{-\frac{s^2}{2}}}{s^2 + (2t-s)^2}\,\mathrm{d}s, \qquad I_1(t) := \int_0^\infty \frac{s\,\mathrm{e}^{-\frac{s^2}{2}}}{s^2 + (2t-s)^2}\,\mathrm{d}s.$$

Set $\tau = \Re t$ and $\theta = \arg t$. If $\theta < \frac{\pi}{6}$, then $1 - 2\tan^2\theta > 1/3$ and

$$|s^2 + (2t-s)^2| > \frac{2}{3}\tau^2, \qquad \forall s \geqslant 0. \tag{A4}$$

If $\mu \in \mathcal{C}_\epsilon(\mu_{\max})$ and $s$ satisfies the hypotheses, then $-\frac{\pi}{6} < \theta \leqslant 0$ and thus $\tau > \frac{\sqrt{3}}{2\sqrt{2}}\frac{1}{\mu_{\max}}|s|$. We then get

$$0 < I_2(t) < 4(\mu_{\max})^2\sqrt{\frac{\pi}{2}}\,|s|^{-2}, \qquad 0 < t\,I_1(t) < 2\sqrt{2}\,\mu_{\max}\,|s|^{-1}. \tag{A5}$$

Since $|s| > \mu_1\sqrt{2}\,|t|$, we get

$$\mu t - 1/2 - 4\pi\frac{\lambda_2^2}{\mu}t\,\sin(t^2) = o(|t|^{-1})$$

$$-1/2 + 4\pi\frac{\lambda_2^2}{\mu}t\cos(t^2) = o(|t|^{-1})$$

which implies

$$4\pi\frac{\lambda_2^2}{\mu}t^2 = \mu^2\,t^2 + \frac{1}{2} - \mu t + o(|t|^{-1}). \tag{A6}$$

But if $t$ is large enough and thus $s$ is large enough, for any $\mu$ satisfying $\mu_1 < |\mu| < \mu_{\max}$, (A6) is false. □

As we covered all the possible regions for the zeros when $\mu$ varies on $\mathcal{C}_\epsilon(\mu_{\max})$ ($f_{1,\mu,+}$ does not have any zero on $\mathbb{R}^-$), these lemmas imply that the set of zeros of $f_{1,\mu}$ and $f_{1,\mu,+}$ is bounded.

Thus, a possible accumulation point is necessarily at a finite distance. This proves proposition A. □

Now it can be shown that the construction of the zeros can be performed beyond the accumulation point. This is due to the fact that this point cannot be a pole of $f_{1,\mu_{\mathrm{lim}}}$ (see [11]), and is thus a regular point, since the branch point has been avoided. It is then possible to apply the local existence theorem again and, finally, there is no obstruction at all. This proves the existence of the zeros for any finite real value of $\mu$.

These zeros are denoted by $s_0(\mu)$ and $s_1(\mu)$. $s_0(1)$ and $s_1(1)$ may be associated with bound states or resonances of a one-dimensional oscillator coupled to a scalar massless boson field via the Hamiltonian $H'_{\mu=1}$, where

$$H'_\mu = a^*a \otimes I + \mu\, I \otimes H_{\mathrm{phot}} + \lambda_2(a^* \otimes c(g_2) + a \otimes (c(g_2))^*). \tag{A7}$$

$g_2$ is even and coincides with $g_1$ on $\mathbb{R}^+$. For $\mu < \mu_c$ (see (29)), $s_0(\mu)$ is an eigenvalue of (A7). This eigenvalue is negative.

## References

[1] Arai A 1981 *J. Math. Phys.* **22** 2539
[2] Arai A 1981 *J. Math. Phys.* **22** 2549
[3] Arai A 1989 *J. Math. Anal. Appl.* **140** 270
[4] Arai A and Hirokawa M 2001 *Rev. Math. Phys.* **13** 513
[5] Bach V, Fröhlich J and Sigal I M 1995 *Lett. Math. Phys.* **34** 183
[6] Bach V, Fröhlich J and Sigal I M 1998 *Adv. Math.* **137** 205
[7] Bach V, Fröhlich J and Sigal I M 1998 *Adv. Math.* **137** 299
[8] Billionnet C 1998 *Ann. Inst. Henri Poincaré* **68** 1
[9] Billionnet C 1998 *J. Phys.* A **31** 623
[10] Billionnet C 2000 *Eur. Phys. J.* D **8** 157
[11] Billionnet C 2001 *Ann. Henri Poincaré* **2** 361
[12] Billionnet C 2000 *Lett. Matt. Phys.* **54** 61
[13] Cohen-Tannoudji C and Avan P 1977 *États atomiques couplés à un continuum. Atomes et molécules hautement excités* (Ed. du C.N.R.S.)
[14] Cohen-Tannoudji C, Dupont-Roc J and Grynberg G 1988 *Processus d'Interaction entre Photons et Atomes* (Paris: Interéditions/Editions du C.N.R.S.) (Engl. transl. New York: Wiley, 1992)
[15] Friedrichs K O 1948 *Comm. Pure Appl. Math.* **1** 361
[16] Moses H E 1973 *Phys. Rev.* A **8** 1710